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Technical Note

Stratified Poiseuille gas flows in horizontal channels

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Abstract

The present short communication shows an exact solution of the Navier–Stokes equations in the case of a channel filled with gas and with temperature contrast between the boundaries. This exact solution is then compared with the result of a numerical simulation made using a numerical code widely used in fire safety engineering. It shows the ability of the code to reproduce this highly stratified flow. Nusselt numbers are then estimated.

Keywords: Navier–Stokes; Exact solution; Stratification; Poiseuille; Channel; Non-Boussinesq

1. Introduction

There are very few exact solutions of the Navier–Stokes equations ([1,8]). The real interest of such exact solutions may be questioned, because they are generally subject to instabilities, sometimes at a relatively low Reynolds number. Moreover, they rarely represent a situation with a practical interest.

However, these exact solutions are useful for making a comparison with a numerical calculation, thus providing some insight on the quality of the simulation tool used.

In the present short communication, we first show that there is an exact solution for gas flows in horizontal channels (of height h) with vertical density gradients which may be very high (leading to a non-Boussinesq situation). The vertical density gradient is created by imposed temperatures on the top and bottom boundaries (see Fig. 1). The flow is considered far downstream of the inlet (say, at a distance L from the entrance, with $L \gg h$, see Fig. 1), so that the precise form of the entrance conditions is unimportant and the gradients in velocity and temperature are purely vertical. This situation is rather different from the one studied in [4], where the flow develops in a gap between two parallel vertical boundaries, gravity being parallel to the main direction of the flow and the small mass flow through each cross section of the gap being small.

In view of the very high density gradients considered, we focus on flows of an ideal gas. The exact solution is then compared with the result of a direct numerical simulation made with a code of widespread use for the assessment of hot air motion with the application of fire safety.

2. Steady laminar parallel flow

2.1. Channel flow equations for an ideal gas at low Mach number

The flow of a fluid of variable density in a 2D channel of height H is considered. The flow is assumed to be parallel and the streamwise direction is denoted by x. The vertical direction is denoted by z, so that

$$\underline{u} = U(z)\underline{e}_x, \quad \rho = \rho(z). \tag{1}$$

The fluid is assumed to be an ideal gas in the low Mach-number limit.

For a fluid of variable density, the Navier–Stokes equations are written as

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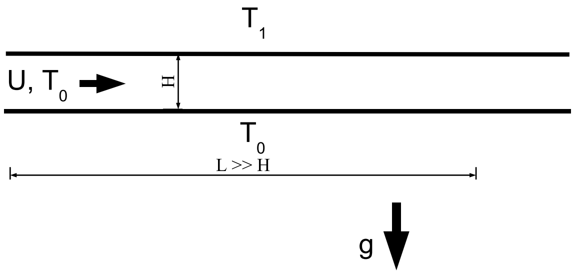

Fig. 1. Stratified flow in a horizontal channel, sketch of the flow and boundary conditions.

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}+g_i$$
$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u_i}{\partial x_i}=0 \qquad (2)$$

where τ_{ij} represents the viscous stress and for a Newtonian fluid is given by

$$\tau_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\mu'\frac{\partial u_k}{\partial x_k}\delta_{ij} \qquad (3)$$

(μ is the dynamic viscosity and $\mu'\approx\mu$). The energy conservation may be expressed through the enthalpy equation, which, for an ideal gas, is written as

$$\frac{\partial \rho C_p T}{\partial t}+\frac{\partial \rho C_p T u_j}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\rho C_p \kappa\frac{\partial T}{\partial x_j}\right) \qquad (4)$$

where T is the temperature, C_p is the specific heat and κ is the diffusivity of heat.

Following an infinitesimal stream-tube, $\mathrm{d}P\approx-\rho\mathrm{d}(u^2/2)\approx\rho(u^2/2)$, and therefore, since in an ideal gas the sound celerity is given by $c=\sqrt{\gamma P/\rho}$, with $\gamma=C_p/C_v$

$$\frac{\mathrm{d}P}{P}\approx\frac{\gamma}{2}\mathrm{d}\left(\frac{u^2}{c^2}\right)\approx\frac{\gamma}{2}M^2 \qquad (5)$$

where M is the Mach number and $\approx$ reads 'of the same order of magitude'. If $M^2\ll 1$, then $\mathrm{d}P/P\ll 1$. The differential form of the state equation of an ideal gas in the present low Mach number situation is therefore

$$\frac{\mathrm{d}\rho}{\rho}+\frac{\mathrm{d}T}{T}\approx 0, \qquad (6)$$

whence

$$\rho T=\rho_0 T_0 \qquad (7)$$

with ρ_0 the density at a reference temperature T_0.

Therefore, the enthalpy equation reduces to

$$\rho_0 C_p T_0\frac{\partial u_j}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\rho C_p\kappa\frac{\partial T}{\partial x_j}\right) \qquad (8)$$

where C_p has been taken as a constant.

2.2. Exact solution of the equations

For a steady laminar parallel flow, the equations (2) reduce to

$$0=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}z}\left(\mu\frac{\mathrm{d}U}{\mathrm{d}z}\right)$$
$$0=-\frac{1}{\rho}\frac{\partial P}{\partial z}-g$$
$$0=\frac{\mathrm{d}}{\mathrm{d}x_j}\left(\rho C_p\kappa\frac{\mathrm{d}T}{\mathrm{d}x_j}\right)$$

with the boundary conditions as described in the introduction and sketched in Fig. 1.

Defining $a=-\frac{\partial P}{\partial x}$, it follows

$$\frac{\mathrm{d}a}{\mathrm{d}z}=-\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial z}\right)=-\frac{\partial}{\partial x}(-\rho g)=0. \qquad (9)$$

Hence, the momentum equation on the x-coordinate becomes

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\mu\frac{\mathrm{d}U}{\mathrm{d}z}\right)s=-a, \qquad (10)$$

which can be integrated to

$$\mu\frac{\mathrm{d}U}{\mathrm{d}z}=-az+\tau_0 \qquad (11)$$

where τ_0 is the shear stress at $z=0$. Note that if $a\neq 0$, $e=\tau/a$ is a height at which $\frac{\mathrm{d}U}{\mathrm{d}z}=0$ ($0<e<H$ exists since $U(0)=U(H)=0$). A good approximation for dry air is a constant Prandtl number $Pr=\frac{\mu}{\rho\kappa}$ and specific heat C_p (see [6], table 5-1-8). Therefore, the heat equation can be integrated to

$$\mu\frac{\mathrm{d}T}{\mathrm{d}z}=\frac{Prq_0}{C_p} \qquad (12)$$

where q_0 is the vertical heat flux at $z=0$.

The variation of μ with temperature in an ideal gas is given by the Sutherland formula:

$$\mu=\mu_{\mathrm{r}}\sqrt{\frac{T}{T_{\mathrm{r}}}}\frac{1+\frac{C}{T_{\mathrm{r}}}}{1+\frac{C}{T}} \qquad (13)$$

with μ_{r} the dynamic viscosity at a given reference temperature T_{r} and C a constant. For dry air, $C=123.6$ K, and at $T_{\mathrm{r}}=273$ K, $\mu_{\mathrm{r}}=17.1\times10^{-6}$ Pa.s. Note that the reference temperature may be chosen arbitrary provided the reference dynamic viscosity is calculated according to this reference temperature. It follows that

$$\frac{\mathrm{d}U}{\mathrm{d}z}=\frac{-az+\tau}{\mu_0}\left[\sqrt{\frac{T}{T_0}}\frac{1+\frac{C}{T_0}}{1+\frac{C}{T}}\right]^{-1} \qquad (14)$$

$$\frac{\mathrm{d}T}{\mathrm{d}z}=\frac{Prq_0}{C_p\mu_0}\left[\sqrt{\frac{T}{T_0}}\frac{1+\frac{C}{T_0}}{1+\frac{C}{T}}\right]^{-1}. \qquad (15)$$

Writing $Z=z/H$, $U_0=\tau H/\mu_0$, $\tilde{u}=U/U_0$, $\alpha=aH/\tau$, $\theta=T/T_0$ (where T_0 is the bottom boundary temperature), $\beta=C/T_0$, and $\gamma=\frac{Prq_0H}{C_p\mu_0T_0(1+\beta)}$, it follows

$$\frac{\sqrt{\theta}}{1+\beta/\theta}\frac{\mathrm{d}\tilde{u}}{\mathrm{d}Z} = -\alpha Z + 1 \tag{16}$$

$$\frac{\sqrt{\theta}}{1+\beta/\theta}\frac{\mathrm{d}\theta}{\mathrm{d}Z} = \gamma. \tag{17}$$

The temperature equation can be rewritten

$$\frac{(\theta/\beta)^{3/2}}{1+(\theta/\beta)}\frac{\mathrm{d}(\theta/\beta)}{\mathrm{d}Z} = \gamma\beta^{-3/2}. \tag{18}$$

Therefore, using $\int \frac{u^{3/2}}{1+u}\mathrm{d}u = 2(u^{3/2}/3 - u^{1/2} + \arctan u^{1/2})$, [1]

$$Z = \frac{2\beta^{3/2}}{\gamma}\left[\frac{\theta^{3/2}-1}{3\beta^{3/2}} - \frac{\theta^{1/2}-1}{\beta^{1/2}} + \arctan\frac{\theta^{1/2}}{\beta^{1/2}} - \arctan\frac{1}{\beta^{1/2}}\right]. \tag{19}$$

Meanwhile, the velocity profile can be obtained from

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\theta} = \frac{\mathrm{d}\tilde{u}}{\mathrm{d}Z}\frac{\mathrm{d}Z}{\mathrm{d}\theta} = \frac{-\alpha Z + 1}{\gamma} \tag{20}$$

with boundary conditions $U(0) = 0$ and $U(H) = 0$. The velocity can be computed from

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\theta} = \frac{1}{\gamma} - \frac{\alpha}{\gamma} \times \frac{2\beta^{3/2}}{\gamma}\left[\frac{\theta^{3/2}-1}{3\beta^{3/2}} - \frac{\theta^{1/2}-1}{\beta^{1/2}} + \arctan\frac{\theta^{1/2}}{\beta^{1/2}} - \arctan\frac{1}{\beta^{1/2}}\right] \tag{21}$$

with $\tilde{u}(0) = 0$ and $\tilde{u}(1) = 0$, which gives the condition to compute α. This equation may be integrated analytically using special functions. However, since the integrand is not stiff, it is simpler and very fast to perform the integration numerically with any method (e.g. Trapezium or Simpson's rule) and to tabulate the resulting function. Another very simple option is to approximate the integrand with a polynomial (e.g. Berstein polynomials, as in [3]). A typical result is given in Fig. 2.

2.3. Comparison with a numerical simulation

The result can be compared with a direct numerical simulation (DNS). The simulation shown here is achieved with the code FDS in its DNS configuration (see [5]). The simulation method used in this code is based on a modified divergence constraint for the calculation of the pressure field in a gas with large density differences (see [5,2]) and, therefore, leads to a good numerical efficiency for low Mach number flows.

The simulation domain is 1 m long and 0.01 m high, with 400 cells in the streamwise direction and 20 in the vertical direction. The simulated case is the same as the one of Fig. 2. The result of the direct numerical simulation is strictly identical to the result of Fig. 2.

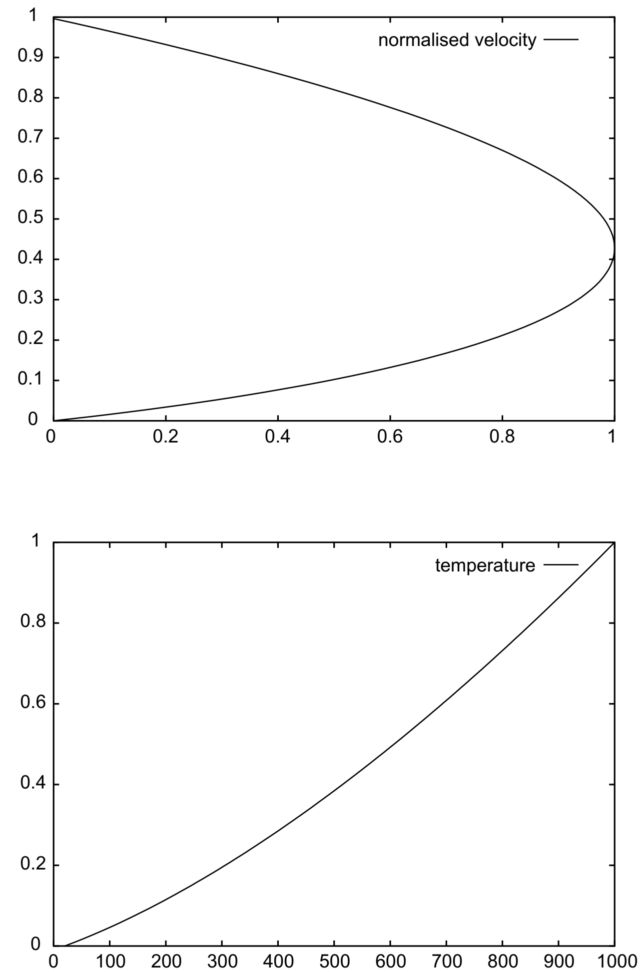

Fig. 2. Velocity and temperature profiles in a channel filed with dry air, for a bottom temperature of 20 °C and a top temperature of 1000 °C (the height of the channel is 0.01 m and the pressure gradient is 0.01 Pa/m; the resulting Reynolds number is 15). The calculation was performed using both the solution of Section 2.2 and DNS (see Section 2.3), and the two solutions superpose exactly on the picture.

3. Vertical heat flux

If $\theta_1 = T_1/T_0$ is the non-dimensional temperature of the top boundary, equation (19) leads to

$$\frac{Prq_0H}{C_p\mu_0T_0(1+\beta)} = 2\beta^{3/2}\left[\frac{\theta_1^{3/2}-1}{3\beta^{3/2}} - \frac{\theta_1^{1/2}-1}{\beta^{1/2}} + \arctan\frac{\theta_1^{1/2}}{\beta^{1/2}} - \arctan\frac{1}{\beta^{1/2}}\right]. \tag{22}$$

Therefore, the heat flux per unit area can be estimated from knowledge of θ_1:

$$q_0 = 2\beta^{3/2}\frac{C_p\mu_0T_0(1+\beta)}{PrH}\left[\frac{\theta_1^{3/2}-1}{3\beta^{3/2}} - \frac{\theta_1^{1/2}-1}{\beta^{1/2}} + \arctan\frac{\theta_1^{1/2}}{\beta^{1/2}} - \arctan\frac{1}{\beta^{1/2}}\right]. \tag{23}$$

If the heat flux is written in an exchange coefficient form

$$q_0 = h(T_1 - T_0), \tag{24}$$

[1] since $\int_a^b \frac{u^{3/2}}{1+u}\mathrm{d}u = 2\int_{a^{1/2}}^{b^{1/2}} \frac{v^4\mathrm{d}v}{1+v^2}$ and $\frac{v^4}{1+v^2} = v^2 - 1 + \frac{1}{1+v^2}$.

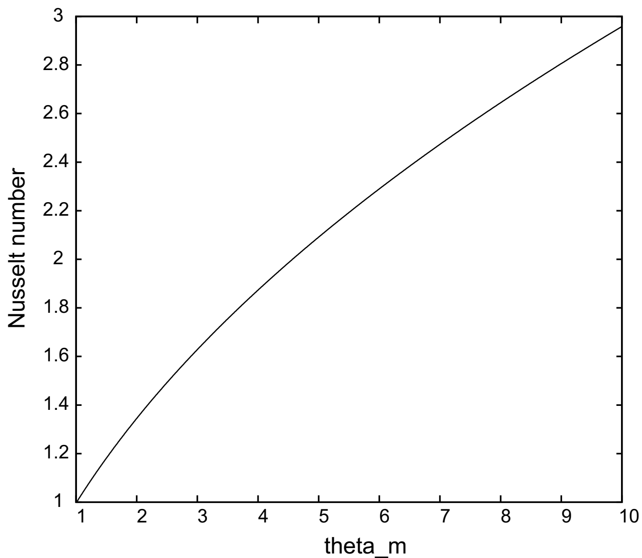

Fig. 3. Nusselt number as a function of θ_1, for dry air (bottom boundary temperature 20°).

one finds

$$h = 2\beta^{3/2}\frac{C_p\mu_0(1+\beta)}{PrH(\theta_1-1)}\left[\frac{\theta_1^{3/2}-1}{3\beta^{3/2}}-\frac{\theta_1^{1/2}-1}{\beta^{1/2}}+\arctan\frac{\theta_1^{1/2}}{\beta^{1/2}}-\arctan\frac{1}{\beta^{1/2}}\right]. \tag{25}$$

Defining the Nusselt number for the present case as

$$Nu = \frac{hH}{\rho_0 C_p \frac{\nu_0}{Pr}}, \tag{26}$$

the above theory leads to

$$Nu = 2\beta^{3/2}\frac{1+\beta}{\theta_1-1}\left[\frac{\theta_1^{3/2}-1}{3\beta^{3/2}}-\frac{\theta_1^{1/2}-1}{\beta^{1/2}}+\arctan\frac{\theta_1^{1/2}}{\beta^{1/2}}-\arctan\frac{1}{\beta^{1/2}}\right]. \tag{27}$$

Fig. 3 shows the value of the Nusselt number for dry air.

4. Case of a Boussinesq flow

Here, it is assumed that temperature differences are small, so that $\theta = 1 + \theta'$ and $\theta' \ll 1$. Equations (19) and (21) can be developed to first order, leading to

$$Z = \frac{\theta'}{\theta'_1} \quad \text{and} \quad \tilde{u} = Z - Z^2 \tag{28}$$

therefore the velocity profile is exactly a Poiseuille isothermal profile. The thermal stratification has no effect on the velocity field. The exchange coefficient in the Boussinesq case reduces to

$$h_{\mathrm{B}} = \frac{C_p\mu_0}{PrH}, \tag{29}$$

corresponding to the Nusselt number $Nu = 1$, as expected.

5. Conclusion

The laminar flow of hot gas in a horizontal channel was considered. It was shown that there exists an exact solution of the Navier–Stokes equations in this case. This solution was compared with a direct simulation using the code FDS, leading to exactly the same result. This shows that the simulation method used in this code, based on a modified divergence constraint for the calculation of the pressure field in gas with large density differences (see [5,2]), is suitable for such a calculation.

The Nusselt number as a function of temperature ratio was calculated.

The solution proposed in this note is an exact solution in the sense that it may be computed in terms of specials functions or from a simple integration of a non-stiff integral. It is given in the form $z = z(T)$, $U = U(T)$, which may be seen, as argued by [4], to be useless for a stability analysis. However, as explained above, the physical situation is in the present case different from the one studied by [4], and the equation $z = z(T)$ forms a very good and smooth diffeomorphism, which may be inverted very simply. As a consequence, it is a better starting point for a stability analysis than a DNS. It must be also be noted that a stability analysis in the present case is mostly interesting in the stable case, when the hot boundary is the top one, creating gravity waves in the flow. This will be a subject of further investigation (in particular to detect preferential wavelength).

It is of interest also to note that the velocity maximum is in the low temperature zone. On the other hand, when a gravity current of hot gas is propagating underneath a ceiling, it is well known that the velocity maximum is in the hot layer (see [7]). Even though it is clear that a gravity current is unsteady, it could be expected that the steady flow is simply a long time limit of the gravity current flow. This point should be investigated in future work.

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